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$$H_n(\eta, a) = \sum \frac{\eta_1^{a_1}}{a_1!} \cdot \frac{\eta_2^{a_2}}{a_2!} \cdots \frac{\eta_k^{a_k}}{a_k!} \frac{\partial^n f}{\partial x_1^{a_1} \partial x_2^{a_2} \cdots \partial x_n^{a_n}}$$

is indefinite at the point (a_1, a_2, \dots, a_k) , the function f has neither a maximum nor a minimum at the point (a_1, a_2, \dots, a_k) . If n is even and $H_n(\eta, a)$ is positive definite the function f has a minimum, while if $H_n(\eta, a)$ is negative definite f has a maximum.

The case, η even and $H_n(\eta, a)$ semi-definite, as usual requires a more elaborate investigation.

PRINCETON, December, 1906.

DIVIDING BY ZERO.

By DR. G. A. MILLER.

Since the seventh century of our era the Hindus have considered division by zero,* and in the twelfth century the noted Hindu mathematician and astronomer Bhaskara gave the rule that the quotient obtained by dividing a number by zero is not changed by adding even a large number to it or by subtracting a large number from it. The same thought is expressed in modern times by "the quotient obtained by dividing a number which is not zero by zero is infinite." This rule is commonly understood to mean, explained by Krishna, a commentator of Bhaskara, that if we divide a given number which is not zero by a number which is small the quotient may be made to exceed any finite number if the divisor is made sufficiently small.

According to this interpretation $a/0$ need not mean the same thing as $\frac{a}{b-b}$. The matter may be made perfectly clear by adopting the notation used by Professor Pierpont in his recent work on the Theory of Functions. If a and b are two distinct numbers then there is an infinity of numbers which do not differ any more from a than b does. The totality of these numbers are said to form the domain of a , whose norm is $a \sim b = \rho$. This totality or aggregate of numbers is denoted by $D_\rho(a)$. It is sometimes desirable to exclude a from its domain. In this case the domain is said to be deleted and it is denoted by $D_\rho^*(a)$. In particular, $D_\rho^*(0)$ means all the numbers which differ from zero by not more than ρ , with the exception of zero itself. By making ρ sufficiently small we obtain the aggregate of numbers which are commonly considered when we think of the meaning of $a/0$,

*Cf. Algebra with arithmetic and mensuration from the Sanscrit of Brahme-gupta and Bhascara translated by H. T. Colebrooke. London, 1817, p. 137.

so that this symbol is merely an abbreviation of the more rational symbol $a/D_p \ast (0)$.

While all are aware that division by zero, which was allowed by some of the most eminent older mathematicians, Euler for example, has caused a great deal of confusion and hence is commonly ruled out in modern mathematics, there are still some such practices in elementary mathematics which are apt to lead to confusion. It is the aim of this note to point to two instances of this kind and to suggest a method of avoiding the difficulties.

The first of these may be illustrated by the equations

$$\frac{1}{x} + \frac{1}{y} = 1, \quad x + y = xy.$$

The second of these is obtained by clearing the first of fractions. No one will question the geometric interpretation of the second as it is a common form of the equation of a hyperbola passing through the origin. This hyperbola contains only one point whose co-ordinates do not satisfy the former of the given equations, viz., the origin. As there is no other point whose co-ordinates satisfy this equation we have the interesting result that the loci of $\frac{1}{x} + \frac{1}{y} = 1$ and $x + y = xy$ differ only with respect to a single point.

If the loci of these two equations were drawn accurately the microscope of highest power would not make it possible to observe any difference since such an instrument could not exhibit the lacuna caused by the missing point. Hence some might at first be inclined to believe that for practical purposes it would make no difference whether we should call these equations equivalent or not. That this is far from the truth follows directly from the solution of such systems of simultaneous equations as

$$\begin{aligned} \frac{1}{x} + \frac{1}{y} = 1, \quad x + y = xy, \\ x = y, \quad x = y. \end{aligned}$$

It is evident that the former of these systems is satisfied by only one pair of values for x and y while the latter is satisfied by two pairs, viz., $(0, 0)$, $(\frac{1}{2}, \frac{1}{2})$. Moreover, the method suggested in the elementary algebras for the solution of such a system as

$$\frac{a}{x} + \frac{b}{y} = 1, \quad \frac{c}{x} + \frac{d}{y} = 1,$$

would not give us a complete solution of the system if the distinction indicated in the preceding paragraph were not observed.

The other point to which we wish to call attention in this connection is that the trigonometric equation $\tan 90^\circ = \infty$ is open to serious objection, for

$\tan A = \frac{\sin A}{\cos A}$, and hence $\tan 90^\circ = \frac{\sin 90^\circ}{\cos 90^\circ} = \frac{1}{0}$; but, as we cannot divide by 0, we must say that $\tan 90^\circ$ has no value unless we assign to it some arbitrary value independently of the fact that $\tan A = \frac{\sin A}{\cos A}$. Suppose that A is equal to some number in $D_\rho * (\frac{1}{2}\pi)$, where ρ represents a small number. As ρ is decreased the smallest possible value of $\tan A$ is increased and we can always select ρ in such a way that $\tan A$ must be larger than any arbitrary number. This fact may be expressed by the equation

$$\tan D_\rho * (\frac{1}{2}\pi) = \infty \text{ when } \rho \doteq 0.$$

It seems very unfortunate that we should tell the student of elementary algebra that it is not permitted to divide by 0 and then in trigonometry tell him that $\tan \frac{\pi}{2} = \frac{1}{0} = \infty$. It appears to be much better to say that $\tan A$ has a meaning for all values of A except when $A = k\frac{1}{2}\pi$. For these special values, it has no meaning but it becomes indefinitely large when A is very nearly equal to a number of the form $k\frac{1}{2}\pi$. It is evident that similar remarks apply to $\cot A$, $\sec A$, and $\csc A$. Trigonometry appears to be in special need of being freed from the hazy terminology which appealed to the Hindu mind of a thousand years ago. The main conclusions which have been reached in the above considerations are:

While we may divide by every number in $D_\rho * (0)$ we are not allowed to divide by every number in $D_\rho (0)$. The two domains differ only with respect to one number and if they could be represented geometrically the microscope of highest power in existence would not reveal the lacuna in $D_\rho * (0)$ caused by the omission of 0. If we multiply by a factor to clear of fractions and then make this factor equal to zero, it is equivalent to dividing by zero and hence this operation cannot be allowed. It is better to say that $a/0$ has no meaning in elementary mathematics than to assign it an arbitrary value since the student cannot appreciate the need of such arbitrary values until he is more mature. There are many to whom the ordinary treatment of the trigonometric functions of $k\frac{1}{2}\pi$ appear objectionable and it is probable that the introduction of the concepts of *domain* and *deleted domain* would tend towards greater clearness in this part of the trigonometry.
